

4-mavzu: To‘la differensial tenglamalar va unga keladigan tenglamalar

Reja:

- 1) To‘la differensial tenglamaning fizik va geometrik mazmuni
- 2) To‘la differensial tenglama va uni tekshirish sharti
- 3) To‘la differensial tenglama
- 4) Misollar yechish

To‘la differensial tenglama

Ta’rif. Agar

$$M(x, y)dx + N(x, y)dy = 0 \quad (1)$$

Tenglamada $M(x, y)$ va $N(x, y)$ funksiyalar uzluksiz, differensiallanuvchi funksiyalar bo‘lib, bular uchun

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad (2)$$

Munosabat o‘rinli bo‘lsa, (1) tenglama to‘la differensialli tenglama deyiladi, bunda $\frac{\partial M}{\partial y}$ va $\frac{\partial N}{\partial x}$ funksiyalar biror sohada uzluksiz funksiyalardir.

To‘la differensialli tenglamalarni integrallash.

Agar (1) tenglamaning chap tomoni to‘la differensial bo‘lsa, u holda (2) shartning bajarilishini va aksincha, (2) shart bajarilsa, (1) tenglamaning chap tomoni biror $u(x, y)$ funksiyaning to‘la differensialli bo‘lishini isbotlaymiz, yoki boshqacha aytganda (1) tenglamaning ko‘rinishi

$$du(x, y) = 0 \quad (3)$$

bo‘lishini va demak, uning umumiy integrali

$$u(x, y) = C$$

bo‘lishini ko‘rsatamiz.

Dastlab, (1) tenglamaning chap tomonini biror $u(x, y)$ funksiyaning to‘la differensialli deb faraz qilaylik, ya’ni

$$M(x, y)dx + N(x, y)dy = du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy;$$

bu holda

$$M = \frac{\partial u}{\partial x}; \quad N = \frac{\partial u}{\partial y}. \quad (4)$$

Birinchi munosabatni y bo'yicha, ikkinchi munosabatni esa x bo'yicha, differensiallab,

$$\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial x \partial y}; \quad \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial y \partial x}$$

tenglikni hosil qilamiz. Ikkinchi tartibli hosilalar uzluksiz deb faraz qilsak,

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

bo'ladi, ya'ni (2) tenglik (1) tenglamaning chap tomoni biror $u(x, y)$ tenglamaning to'la differensial bo'lishining zaruriy shartidan iboratdir. Bu shartning yetarli shart bo'lishini ham, ya'ni (2) tenglik bajarilganda (1) tenglamaning chap tomoni biror $u(x, y)$ funksiyaning to'la differensial bo'lishini ko'rsatamiz.

$$\frac{\partial u}{\partial x} = M(x, y)$$

munosabatdan

$$u = \int_{x_0}^x M(x, y) dx + \varphi(y)$$

ni topamiz, bunda x_0 yechim mavjud bo'lgan sohadagi ixtiyoriy nuqtaning absissasi.

x bo'yicha integrallashda y ni o'zgarmas miqdor deb hisoblaymiz va shuning uchun integrallashda hosil bo'lgan ixtiyoriy o'zgarmas miqdor y ga bog'liq bo'lishi mumkin. $\varphi(y)$ ni (4) munosabatlardan ikkinchisi bajariladigan qilib tanlab olamiz. Buning uchun keying tenglamaning ikkala tomonini y bo'yicha differensiallaymiz va natijani $N(x, y)$ ga tenglaymiz:

$$\frac{\partial u}{\partial y} = \int_{x_0}^x \frac{\partial M}{\partial y} dx + \varphi'(y) = N(x, y);$$

ammo $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ bo'lgani uchun quyidagilarni yoza olamiz:

$$\int_{x_0}^x \frac{\partial N}{\partial x} dx + \varphi'(y) = N, \quad \text{ya'ni} \quad N(x, y)|_{x_0}^x + \varphi'(y) = N(x, y),$$

Yoki

$$N(x, y) - N(x_0, y) + \varphi'(y) = N(x, y).$$

demak,

$$\varphi'(y) = N(x_0, y),$$

Yoki

$$\varphi(y) = \int_{y_0}^y N(x_0, y) dy + C_1.$$

shunday qilib, $u(x, y)$ funksiya

$$u = \int_{x_0}^x M(x, y) dx + \int_{y_0}^y N(x_0, y) dy + C_1$$

Ko'rinishda bo'ladi. Bunda $P(x_0, y_0)$ nuqta atrofida (1) differensial tenglamaning yechimi mavjud bo'lgan nuqtadir.

Bu ifodani ixtiyoriy C o'zgarmas miqdorga tenglab, (1) tenglamaning umumiy integralini hosil qilamiz:

$$\int_{x_0}^x M(x, y) dx + \int_{y_0}^y N(x_0, y) dy = C. \quad (5)$$

1-misol.

$$\frac{2x}{y^3} dx + \frac{y^2 - 3x^2}{y^4} dy = 0$$

Tenglama berilgan. Bu tenglama to'la differensialli tenglama bo'lish yo bo'lmasligini tekshiramiz.

Bu yerda

$$M = \frac{2x}{y^3}; \quad N = \frac{y^2 - 3x^2}{y^4}$$

Deb olamiz, bu holda

$$\frac{\partial M}{\partial y} = -\frac{6x}{y^4}; \quad \frac{\partial N}{\partial x} = -\frac{6x}{y^4}$$

$y \neq 0$ bo'lganda (2) shart bajariladi. Demak, berilgan tenglamaning chap tomoni biror noma'lum $u(x, y)$ funksiyaning to'liq differensial bo'ladi. Bu funksiyani

topamiz. $\frac{\partial u}{\partial x} = \frac{2x}{y^3}$ bo'lgani sababli

$$u = \int \frac{2x}{y^3} dx + \varphi(y) = \frac{x^2}{y^3} + \varphi(y)^3$$

bunda $\varphi(y)$ funksiya y ning hozircha noma'lum bo'lgan funksiyasi. Bu munosabatni y bo'yicha differensiallab va

$$\frac{\partial u}{\partial x} = N = \frac{y^2 - 3x^2}{y^4}$$

tenglikni e'tiborga olib,

$$-\frac{3x^2}{y^4} + \varphi'(y) = \frac{y^2 - 3x^2}{y^4}$$

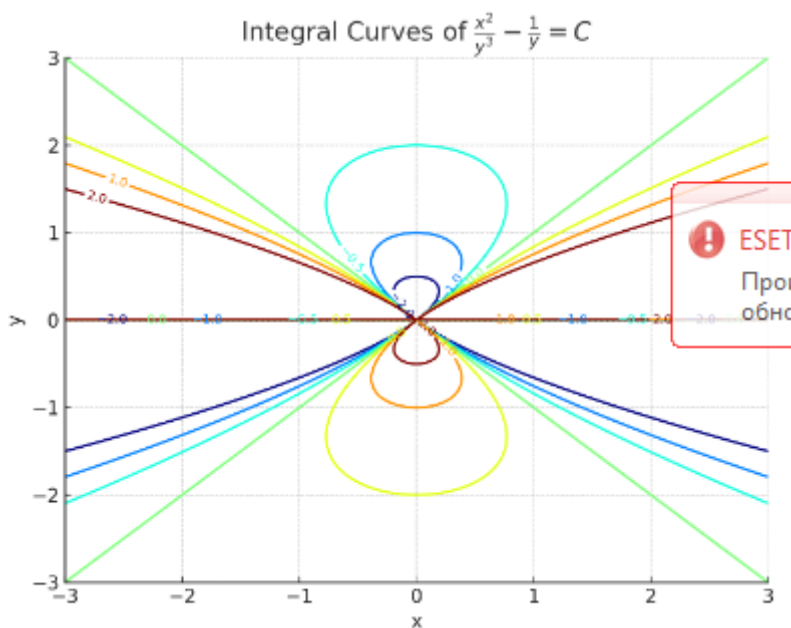
Bo'lishini topamiz. Demak,

$$\varphi'(y) = \frac{1}{y^2}, \quad \varphi(y) = -\frac{1}{y} + C_1, \quad u(x, y) = \frac{x^2}{y^3} - \frac{1}{y} + C_1$$

Shunday qilib,

$$\frac{x^2}{y^3} - \frac{1}{y} = C$$

dastlabki tenglamaning umumiy integralidir.



2-misol. Differensial tenglamani yeching:

$$(\sin xy + xy \cos xy)dx + x^2 \cos xy dy = 0.$$

Yechish. Berilgan tenglama to'liq differensialli tenglama ekanligini tershiramiz:

$$\begin{aligned}\frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}(\sin xy + xy \cos xy) = \\ &= x \cos xy + x \cos xy - x^2 y \sin xy = 2x \cos xy - x^2 y \sin xy,\end{aligned}$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(x^2 \cos xy) = 2x \cos xy - x^2 y \sin xy.$$

Demak, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ shart bajariladi. Shunday qilib, berilgan tenglama to'liq

differensialli tenglama va

$$M = \frac{\partial u}{\partial x} = \sin xy + xy \cos xy, \quad N = \frac{\partial u}{\partial y} = x^2 \cos xy$$

teng. Shuning uchun

$$u(x, y) = \int (\sin xy + xy \cos xy) dx + \varphi(y).$$

Bu yyerda $\varphi(y)$ hozircha noma'lum funksiya.

Integrallab

$$u(x, y) = x \sin xy + \varphi(y)$$

topamiz. Topilgan $u(x, y)$ funksiyaning $\frac{\partial u}{\partial y}$ xususiy hosilasi $x^2 \cos xy$ teng bo'lishi

kerak. Shuning uchun

$$x^2 \cos xy + \varphi'(y) = x^2 \cos xy$$

munosabat o'rinli bo'lishi uchun $\varphi'(y) = 0$ bo'lishi kerak. Bundan esa $\varphi(y) = C$ ekanligi kelib chiqadi. Demak,

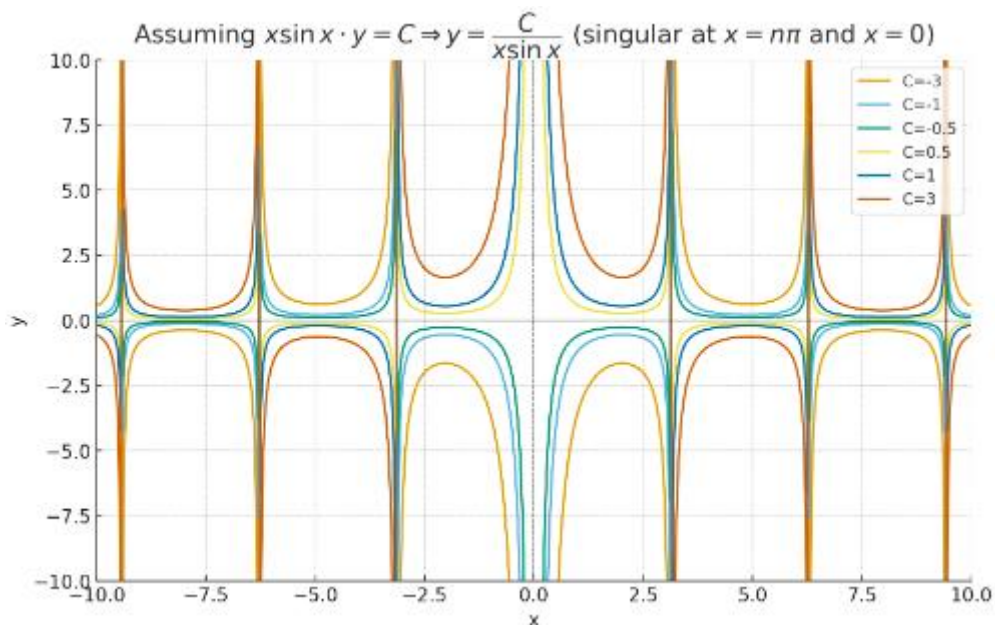
$$u(x, y) = x \sin xy + C.$$

Berilgan differensial tenglama

$$x \sin xy = C$$

umumiy integralga ega.

Ba'zi differensial tenglamalarni integrallashda hadlarni shunday guruhlash mumkinki, osongina integrallanadigan kombinatsiyalarni olish mumkin.



3-misol. Differensial tenglamani yeching:

$$(x^3 + xy^2)dx + (x^2y + y^3)dy = 0. \quad (1)$$

Yechish. $\frac{\partial M}{\partial x} = 2xy$, $\frac{\partial N}{\partial y} = 2xy$ (2) munosabat bajarilganligi uchun berilgan

tenglama to'liq differensilli tenglama bo'ladi. Bu tenglamani guruhlash yo'li bilan osongina $du = 0$ ko'rinishga keiltirish mumkin. Shu maqsadda bu tenglamani quyidagi ko'rinishda yozib ilamiz:

$$x^3 dx + xy(ydx + xdy) + y^3 dy = 0.$$

Differensial qoidalariga ko'ra

$$2. \quad x^3 dx = d\left(\frac{x^4}{4}\right), \quad xy(ydx + xdy) = xy d(xy) = d\left(\frac{(xy)^2}{2}\right), \quad y^3 dy = d\left(\frac{y^4}{4}\right)$$

munosabatlar o'rinli. Shuning uchun (4) tenglamani

$$d\left(\frac{x^4}{4}\right) + d\left(\frac{(xy)^2}{2}\right) + d\left(\frac{y^4}{4}\right) = 0$$

yoki

$$d\left[\frac{x^4}{4} + \frac{(xy)^2}{2} + \frac{y^4}{4}\right] = 0$$

ko'rinishda yozish mumkin. Demak,

$$x^4 + 2(xy)^2 + y^4 = C$$

(1) tenglamaning umumiy integrali bo'ladi.

Misollar

1. $2xydx + (x^2 - y^2)dy = 0$ tenglamani to'liq differensialli ekanligini tekshiring va uni integrallang.

2. $(2 - 9xy^2)xdx + (4y^2 - 6x^3)ydy = 0$ tenglamani to'liq differensialli ekanligini tekshiring va uni integrallang.

3. $e^{-y}dx - (2y + xe^{-y})dy = 0$ tenglamani to'liq differensialli ekanligini tekshiring va uni integrallang.

4. $\frac{y}{x}dx + (y^3 + \ln x)dy = 0$ tenglamani to'liq differensialli ekanligini tekshiring va uni integrallang.

5. $\frac{y}{x}dx + (y^3 + \ln x)dy = 0$ tenglamani to'liq differensialli ekanligini tekshiring va uni integrallang.

6. $\frac{3x^2+y^2}{y^2}dx - \frac{2x^3+5y}{y^3}dy = 0$ tenglamani to'liq differensialli ekanligini tekshiring va uni integrallang.

Savollar

1. To'la differensial tenglamaning fizik ma'nosi nimada?
2. Geometrik nuqtai nazardan to'la differensial tenglama nimani bildiradi?
3. Quyidagi tenglama to'la differensialmi? Tekshirish shartini qo'llang.
4. tenglama to'la bo'lishi uchun qanday shart bajarilishi kerak?
5. Potensial funksiyani topish nima degani?
6. Quyidagi tenglama uchun potensial funksiyani toping.