7-mavzu: Amaliy mashg’ulot ishlanmasi

Hosilaga nisbatan yechilmagan tenglamani yechish usullari

Hosilaga nisbatan yechilmagan differensial tenglama umumiy ko‘rinishda quyidagicha yoziladi:

$$F(x, y, y') = 0$$

Agar uni y′=f(x,y) shaklida ifodalab bo‘lmasa, bunday tenglamalar **hosilaga nisbatan yechilmagan** deyiladi. Ularni quyidagi usullar yordamida yechish mumkin:

- Hosilani ifodalab klassik metodlarga keltirish (ajratiluvchi, chiziqli);
- Parametrli yechimlar topish;
- Yashirin (implicit) yechimlar orqali ifodalash;
- Grafik (yo‘nalishlar maydoni) yordamida yechimni tasavvur qilish.

1-topshiriq. Hosilani ifodalab yechish

Berilgan tenglama:

$$y - xy' + \sqrt{1 + (y')^2} = 0$$

a) Hosila y′y'y′ ni ifodalang (agar mumkin bo‘lsa)
b) Yechimni grafik jihatdan tavsiflang
c) Yo‘nalishlar maydoni ni chizing.

2-topshiriq. Ajratiluvchi tenglamaga keltirish

$$\frac{dy}{dx} = \frac{x}{y}$$

a) O‘zgaruvchilarni ajrating
b) Ikkala tomonni integrallang
c) Yechimni yashirin shaklda yozing

3-topshiriq. Parametrli yechimdan foydalanish

Parametrli ko‘rinishda berilgan egri chiziq:

$$x = t^2 + 1, \quad y = t^3 - t$$

a) $\frac{dy}{dx}$ ni toping
b) y′ ni faqat t orqali ifodalang
c) Egri chiziqning harakat yo‘nalishini grafikda tasvirlang

4-topshiriq. To‘liq differensial tenglama

Tenglama:

$$(2xy + y^2)dx + (x^2 + 2xy)dy = 0$$

a) Tenglama to‘liq differensialmi? Tekshiring.
b) Potensial funksiya F(x,y) ni toping
c) Yechimni yashirin ko‘rinishda yozing

5-topshiriq. Fizik model asosida differensial tenglama tuzing

Vazifa: Havo qarshiligida to‘xtayotgan jism uchun differensial tenglama tuzing. Nyuton II qonuniga asosan:

$$m\frac{dv}{dt} = -kv^2$$

a) Tenglamani hosiladan yeching
b) Yechimni umumiy ko‘rinishda yozing
c) Boshlang‘ich shart: $v(0) = v_0$ ni hisobga oling

1-topshiriq. Hosilaga nisbatan yechilgan va yechilmagan tenglamani ajrating

Quyidagi tenglamalardan qaysi biri hosilaga nisbatan yechilgan, qaysi biri yechilmaganligini aniqlang va izohlang:

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a) $\frac{dy}{dx} = x + y$
b) $xy' + y = \cos x$
c) $y - xy' + \sqrt{1 + (y')^2} = 0$

4-topshiriq. To‘liq differensial tenglama bo‘yicha ishlang

Quyidagi tenglama berilgan:

$$(2xy + y^2)dx + (x^2 + 2xy)dy = 0$$

a) To‘liqlik shartini tekshiring
b) Potensial funksiya F(x,y) ni toping
c) Yechimni F(x,y)=C shaklida yozing

6-topshiriq. Grafik yondashuv (yo‘nalishlar maydoni)

Quyidagi tenglama uchun grafik yo‘nalishlar maydonini (slope field) chizing:

$$\frac{dy}{dx} = x - y$$

a) (0,0), (1,0), (0,1) nuqtalarda tangens yo‘nalishini belgilang
b) Umumiy integral chiziqlarning yo‘nalishini tavsiflang

Uyga vazifalar (yechish uchun):

1. Tenglamani hosilaga nisbatan yeching:

$$xy' + y = \sin(x)$$

2. Quyidagi to‘liq differensial tenglama:
   $(y\cos x + x)dx + (\sin x + 2y)dy = 0$
3. Aylana trayektoriyasining parametrli tenglamasini yozing:
   $x^2 + y^2 = R^2 \Rightarrow x = R\cos t,\ y = R\sin t$

Nazorat savollari:

1. Hosilaga nisbatan yechilmagan differensial tenglama deganda nimani tushunasiz?
2. Koshi masalasi qachon yagona yechimga ega bo‘ladi?
3. Parametrli yechimlar nima uchun ishlatiladi?
4. Yashirin yechimlar qanday aniqlanadi?
5. Quyidagi tenglama qanday usul bilan yechiladi:

$$\frac{dy}{dx} = y \cdot \tan x$$

1-misol. Berilgan kesmada differensial tenglamaning yaqin (approksimatsion) yechimi xatosini baholang:

$$y' = \frac{x}{4} - \frac{1}{1+y^2}, \qquad y(0) = 1;$$

$$y'' = 1 - \frac{x}{2}, \qquad |x| \le \tfrac{1}{2}.$$

Masalaning yechimi

Berilgan tenglama:

$$y' = \frac{x}{4} - \frac{1}{1+y^2}, \quad y(0) = 1; \quad \tilde{y} = 1 - \frac{x}{2}, \quad |x| \le \tfrac{1}{2}.$$

1. **Tenglama o‘ng tomoni**
   O‘ng tomoni
   $f(x,y) = \frac{x}{4} - \frac{1}{1+y^2}$

uzluksiz, chunki x va y bo‘yicha (berilgan oraliqda) to‘liq aniqlangan va differensiallanuvchi.

Hosila:

$$\frac{\partial f}{\partial y} = \frac{2y}{(1+y^2)^2}.$$

Baholash:

$$\left|\frac{\partial f}{\partial y}\right| = \frac{2|y|}{(1+y^2)^2} \le \frac{2|y|}{1+y^2} \le 1.$$

Demak, Lipschitz konstantasi sifatida K=1 ni olish mumkin.

2. Yaqin yechim va haqiqiy yechim orasidagi farqni baholash

Bizda:

$$\left|\frac{dy}{dt} - f(t,y)\right| \le \varepsilon, \quad |y(0) - \tilde{y}(0)| \le \delta.$$

Endi

$$\tilde{y}' = -\tfrac{1}{2},$$

$$f(x,\tilde{y}) = \frac{x}{4} - \frac{1}{1+(1-x/2)^2}.$$

Shu sababli, xato bahosi:

$$|\tilde{y}' - f(x,\tilde{y})| = \left| -\frac{1}{2} - \left(\frac{x}{4} - \frac{1}{1+(1-x/2)^2} \right) \right| \le \frac{1}{64}.$$

Demak,

$$\varepsilon = \tfrac{1}{64}.$$

Boshlang‘ich qiymat mosligi tufayli δ=0

3. Xato formulasidan foydalanamiz
Grönvall tengligiga ko‘ra:

$$\|y(x) - \tilde{y}(x)\| \le \frac{\varepsilon}{K}\left(e^{K|x|} - 1\right).$$

Bu yerda

$$K = 1, \varepsilon = \tfrac{1}{64}.$$

Shunday qilib:

$$|y(x) - \tilde{y}(x)| \le \frac{1}{64}(e^{|x|} - 1).$$

2. Maksimal xato bahosi

$$|x| \le \tfrac{1}{2}$$

bo‘lgani uchun:

$$|y(x) - \tilde{y}(x)| \le \frac{1}{64}(e^{1/2} - 1) < 0.011.$$

Xulosa:
Yaqinlashgan yechim

$$\tilde{y}(x) = 1 - \frac{x}{2}$$

ning haqiqiy yechimdan maksimal og‘ishi

$$|y(x) - \tilde{y}(x)| < 0.011, \quad |x| \le \tfrac{1}{2}.$$

2-misol.

Berilgan kesmada differensial tenglamaning yaqin yechimi xatosini baholang

$$y'' - x^2 y = 0, \quad y(0) = 1,\ y'(0) = 0;$$
$$\tilde{y}(x) = e^{x^4/12}, \quad |x| \le 0.5.$$

Misolning yechimi

Berilgan tenglama:

$$y'' - x^2 y = 0, \quad y(0) = 1,\ y'(0) = 0; \quad \tilde{y}(x) = e^{\frac{x^4}{12}}, \quad |x| \le 0.5.$$

1. Ikkinchi tartibli tenglamani birinchi tartibli sistema ko‘rinishiga keltiramiz:

O‘zgartirish kiritamiz:

$$x_1 = y, \quad x_2 = y',$$

Unda

$$x_1' = x_2, \quad x_2' = x^2 x_1.$$

Yaqin yechim:

$$\tilde{x}_1 = e^{x^4/12}, \quad \tilde{x}_2 = \frac{x^3}{3} e^{x^4/12}.$$

2. Normani aniqlaymiz:

$$\|x\| = |x_1| + |x_2|.$$

Shunda baholash uchun:

$$\left\| \frac{d\tilde{x}}{dt} - f(t, \tilde{x}) \right\| \le \varepsilon, \quad \|y(0) - \tilde{y}(0)\| \le \delta.$$

3. Farqni hisoblaymiz:

$$\left\| \frac{d\tilde{x}}{dt} - f(t, \tilde{x}) \right\| = |\tilde{x}_1' - f_1(t, \tilde{x})| + |\tilde{x}_2' - f_2(t, \tilde{x})|.$$

Ma’lumki:

$$\tilde{x}_1' = \frac{x^3}{3} e^{x^4/12}, \quad f_1 = \tilde{x}_2 = \frac{x^3}{3} e^{x^4/12} \quad \Rightarrow \quad |\tilde{x}_1' - f_1| = 0.$$

$$\tilde{x}_2' = e^{x^4/12} \left(x^2 + \tfrac{x^6}{9} \right), \quad f_2 = x^2 \tilde{x}_1 = x^2 e^{x^4/12}.$$

Farqi

$$|\tilde{x}_2' - f_2| = e^{x^4/12} \cdot \frac{x^6}{9}.$$

4. Maksimal qiymatini baholaymiz:

$$|\tilde{x}_2' - f_2| \le \max_{|x| \le 0.5} \frac{x^6}{9} e^{x^4/12}.$$

$$\le \frac{(0.5)^6}{9} e^{(0.5)^4/12} \approx 0.0017.$$

Shunday qilib,

$$\varepsilon = 0.0017, \quad \delta = 0.$$

Xulosa:
Yaqin yechim

$$\tilde{y}(x) = e^{x^4/12}$$

yuqoridagining haqiqiy yechimdan maksimal og‘ishi

$$|y(x) - \tilde{y}(x)| \le 0.0017, \quad |x| \le 0.5.$$

Misolning davomiy yechimi

Bizda hosilalar:

$$\frac{\partial f_1}{\partial x_1} = 0, \quad \frac{\partial f_1}{\partial x_2} = 1, \quad \frac{\partial f_2}{\partial x_1} = t^2, \quad \frac{\partial f_2}{\partial x_2} = 0.$$

Demak, Lipschitz konstantasi quyidagicha olinadi:

$$K = 2 \cdot \max\{1,\ t^2\} = 2, \quad (|t| \le 0.5).$$

Ma’lum qiymatlar:

$$\varepsilon = 0.0017,\ \delta = 0,\ K = 2.$$

Shunday qilib,

$$\|x(t) - \tilde{x}(t)\| \le \frac{0.0017}{2}\,(e^{2|t|} - 1).$$

Agar |t|≤0.5 bo‘lsa:

$$\|x(t) - \tilde{x}(t)\| \le 0.00085\,(e^1 - 1) < 0.00085 \cdot 1.718 < 0.00146 < 0.002.$$

Natijada:

$$\|x(t) - \tilde{x}(t)\| < 0.002,$$

ayniqsa:

$$|x_1 - \tilde{x}_1| < 0.002.$$