12-mavzu: Kirxgof qonunlari asosida RLC elektr zanjiri modellari (tok va kuchlanishning sistemasi)

Reja:

- Kirxgof qonunlari haqida umumiy tushuncha
- R, L, C elementlarning fizik va matematik modellari
- Kirxgof qonunlari asosida tenglamalar sistemasi tuzish
- O‘zgarmas va o‘zgaruvchan kuchlanishli manbalar uchun modellar
- Amaliy misollar va grafiklar bilan izohlash

1. Kirxgof qonnlari haqida umumiy tushuncha

Kirxgofning elektr zanjirlar nazariyasidagi ikkita asosiy qonuni mavjud:

1-qonun (tugunlar qonuni):

Har qanday tugunda kirayotgan toklarning algebraik yig‘indisi nolga teng:

$$\sum I_{\text{kiruvchi}} = \sum I_{\text{chiquvchi}} \quad \Rightarrow \quad \sum I = 0$$

2-qonun (kontur qonuni):

Zanjirdagi har qanday yopiq kontur bo‘ylab kuchlanishlar algebraik yig‘indisi nolga teng:

$$\sum \mathcal{E} - \sum U = 0$$

Bu qonun asosida tok va kuchlanishlar orasidagi differensial tenglamalar tuziladi.

2. R, L, C lementlarning matematik modellari

Element	Belgilanish	Tenglama	Tavsifi
Rezistor(R)	R	$U_R = IR$	Om qonuni
Induktor(L)	L	$U_L = L\frac{dI}{dt}$	O‘zgaruvchan tokda EMK induksiya
Kondensator(C)	C	$U_C = \frac{1}{C}\int Idt \quad yoki \quad I_C = C\frac{dU}{dt}$	Zaryad to‘planish modeli

3. Kirxgof qonunlari asosida tenglamalar sistemasi tuzish

Oddiy RLC konturi (ketma-ket ulangan):

Zanjir quyidagi elementlardan iborat:
Kuchlanish manbai E(t), rezistor R, induktor L, kondensator C

Kirxgofning 2-qonuni asosida quyidagicha differensial tenglama yoziladi:

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = \mathcal{E}(t)$$

bu yerda:

$q(t)$ — zaryad (kulon),

$I(t) = \frac{dq}{dt}$ — tok.

Agar $\mathcal{E}(t) = 0$ bo‘lsa, bir jinsli tenglama:

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 0$$

Tok orqali ifodalash:

Tokning differensial tenglamasi:

$$L\frac{dI}{dt} + RI + \frac{1}{C}\int Idt = \mathcal{E}(t)$$

Yoki zaryad orqali:

$$Lq'' + Rq' + \frac{1}{C}q = \mathcal{E}(t)$$

4. Maxsus hollardagi RLC modellar

a)O‘zgarmas kuchlanish manbai $(\mathcal{E} = \text{const})$

Turg‘un holat oxirida:

- Induktor ochiq zanjirga teng: $I_L = 0$
- Kondensator — zaryadlangan va tok o‘tmaydi

b) Garmonik kuchlanish manbai: $\mathcal{E}(t) = E_0 \sin \omega t$

Bu holat uchun yechim sinusoidal bo‘ladi, ya’ni:

$$q(t) = A \sin(\omega t + \phi)$$

Zanjirda rezonans holati yuzaga kelishi mumkin, agar:

$$\omega = \omega_0 = \frac{1}{\sqrt{LC}}$$

5. Amaliy misol

Misol: Quyidagi RLC zanjirda kuchlanish manbai $\mathcal{E}(t) = 0$,

boshlang‘ich zaryad $q(0) = q_0$, tok $I(0) = 0$.

Berilgan $R = 2\,\Omega,\ L = 1\,H,\ C = 0.25\,F$

Yechim uchun tenglama:

$$q'' + 2q' + 4q = 0$$

Xarakteristik tenglama:

$$r^2 + 2r + 4 = 0 \quad \Rightarrow \quad r = -1 \pm i\sqrt{3}$$

Umumiy yechim:

$$q(t) = e^{-t}(A\cos(\sqrt{3}t) + B\sin(\sqrt{3}t))$$

Tok:

$$I(t) = \frac{dq}{dt}$$

$$I(t) = \frac{dq}{dt} = \frac{d}{dt}\left[e^{-t}\left(2\cos(\sqrt{3}t) + \frac{2}{\sqrt{3}}\sin(\sqrt{3}t)\right)\right]$$

Bu – **ko‘paytmaning hosilasi** formulasi bo‘yicha:

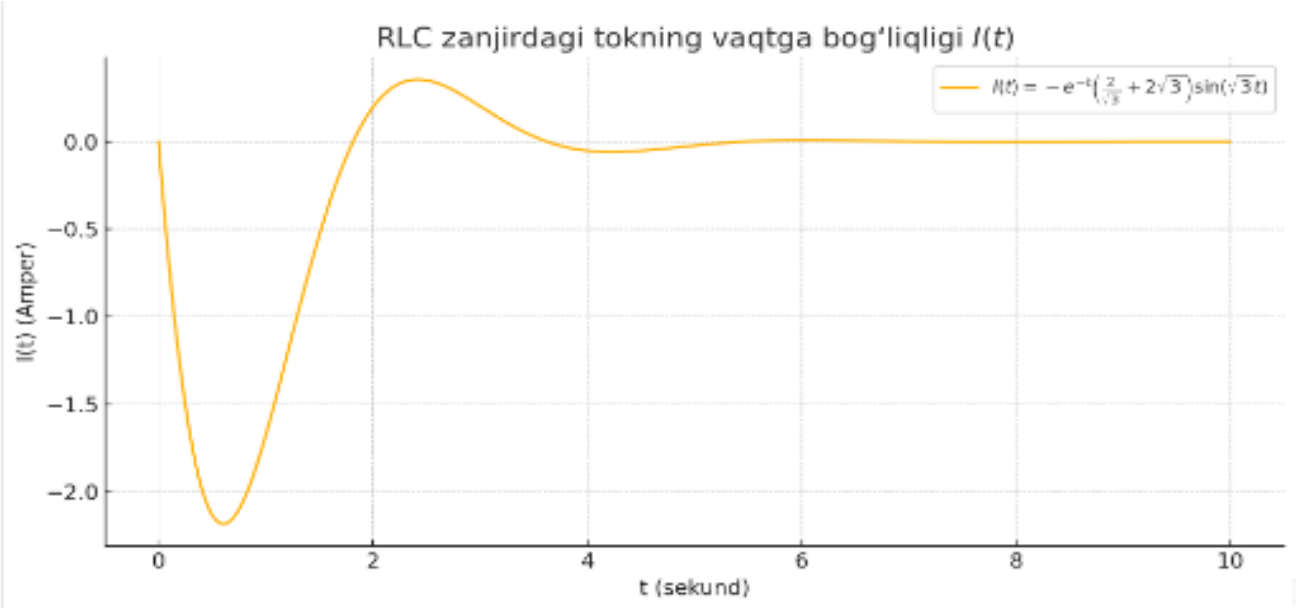
$$I(t) = -e^{-t}\left(2\cos(\sqrt{3}t) + \frac{2}{\sqrt{3}}\sin(\sqrt{3}t)\right) + e^{-t}\cdot\left(-2\sqrt{3}\sin(\sqrt{3}t) + \frac{2}{\sqrt{3}}\cdot\sqrt{3}\cos(\sqrt{3}t)\right)$$

Soddalashtiramiz:

$$I(t) = e^{-t}\left[-\left(2\cos(\sqrt{3}t) + \frac{2}{\sqrt{3}}\sin(\sqrt{3}t)\right) + \left(-2\sqrt{3}\sin(\sqrt{3}t) + 2\cos(\sqrt{3}t)\right)\right]$$

$$I(t) = e^{-t}\left[-\frac{2}{\sqrt{3}}\sin(\sqrt{3}t) - 2\sqrt{3}\sin(\sqrt{3}t)\right]$$

$$I(t) = -e^{-t}\left(\frac{2}{\sqrt{3}} + 2\sqrt{3}\right)\sin(\sqrt{3}t)$$



Yuqoridagi grafikda RLC elektr zanjiridagi tokning vaqt bo‘yicha o‘zgarishi — ya’ni

$$I(t) = -e^{-t}\left(\frac{2}{\sqrt{3}} + 2\sqrt{3}\right)\sin(\sqrt{3}t)$$

funksiyasi tasvirlangan.

Grafik tahlili:

Ekspotensial pasayish e^{-t} tufayli tok tebranishlari vaqt o‘tishi bilan sustlashadi

Grafik **sinusoida** shaklida tebranadi, lekin amplitudasi kamayib boradi — bu **dempferlangan (so‘nuvchi) tebranish** holatidir.

Tok boshlanishda salbiy qiymatdan boshlanadi, bu boshlang‘ich zaryad va induktor ta’siri natijasidir.

1-misol. Bir jinsli RLC zanjir:

Berilgan

$$R = 2\,\Omega,\ L = 1\,H,\ C = 0.25\,F,\ \mathcal{E}(t) = 0$$

Dastlabki shartlar: $q(0) = 2,\ I(0) = 0$

Yechim:

1)Tenglama: $q'' + 2q' + 4q = 0$

2)Xarakteristik tenglama: $r^2 + 2r + 4 = 0 \quad \Rightarrow \quad r = -1 \pm i\sqrt{3}$

3)Umumiy yechim: $q(t) = e^{-t}\left(A\cos(\sqrt{3}t) + B\sin(\sqrt{3}t)\right)$

4)Dastlabki shartlardan:

$$A = 2, \quad B = \frac{2}{\sqrt{3}} \quad \Rightarrow \quad q(t) = e^{-t}\left(2\cos(\sqrt{3}t) + \frac{2}{\sqrt{3}}\sin(\sqrt{3}t)\right)$$

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5)Tok:

$$I(t) = \frac{dq}{dt} = -e^{-t}\left(2\cos(\sqrt{3}t) + \frac{2}{\sqrt{3}}\sin(\sqrt{3}t)\right) + e^{-t}\left(-2\sqrt{3}\sin(\sqrt{3}t) + 2\cos(\sqrt{3}t)\right)$$

6)Soddalashtirilgan tok:

$$I(t) = -e^{-t}\left(\frac{2}{\sqrt{3}} + 2\sqrt{3}\right)\sin(\sqrt{3}t)$$

Misol 2. Bir jinsli bo‘lmagan tenglama (garmonik manba):

Berilgan: $R = 1\,\Omega, L = 1\,H, C = 1\,F, \mathcal{E}(t) = 5\cos t$

Tenglama: $q'' + q' + q = 5\cos t$

Bu **ikkinchi tartibli chiziqli nojinsli differensial tenglama** bo‘lib, ikki bosqichda yechiladi:

1-bosqich: Bir jinsli tenglama yechimi

$$q'' + q' + q = 0$$

Xarakteristik tenglama:

$$r^2 + r + 1 = 0 \quad \Rightarrow \quad r = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$$

Kompleks ildizlar bo‘lgani uchun yechim:

$$q_{\text{umumiy}}(t) = e^{-t/2}\left(C_1 \cos\left(\frac{\sqrt{3}}{2}t\right) + C_2 \sin\left(\frac{\sqrt{3}}{2}t\right)\right)$$

2-bosqich: Xususiy yechim

Nojinsli qo‘zg‘atuvchi: 5cost

Tajriba funksiyasi sifatida quyidagini tanlaymiz:

$$q_{\text{xususiy}}(t) = A\cos t + B\sin t$$

Hosilalarni hisoblaymiz:

$$q' = -A\sin t + B\cos t, \quad q'' = -A\cos t - B\sin t$$

Endi tenglamaga qo‘yamiz:

$$q'' + q' + q = (-A\cos t - B\sin t) + (-A\sin t + B\cos t) + (A\cos t + B\sin t)$$

Tenglamani soddalashtiramiz:

$$(-A\cos t + B\cos t + A\cos t) + (-B\sin t - A\sin t + B\sin t) = B\cos t - A\sin t$$

Demak:

$$B\cos t - A\sin t = 5\cos t$$

Bu tenglikdan quyidagilar kelib chiqadi: $B = 5, \quad A = 0$

Xususiy yechim:

$$q_{\text{xususiy}}(t) = 5\sin t$$

Yakuniy yechim:

$$q(t) = q_{\text{umumiy}} + q_{\text{xususiy}} = e^{-t/2}\left(C_1 \cos\left(\frac{\sqrt{3}}{2}t\right) + C_2 \sin\left(\frac{\sqrt{3}}{2}t\right)\right) + 5\sin t$$

Yuqoridagi grafikda q(t) — zaryadning vaqtga bog‘liqligi tasvirlangan:

$$q(t) = e^{-t/2}\cos\left(\frac{\sqrt{3}}{2}t\right) + 5\sin t$$

Nazorat savollari:

- Kirxgofning 1- va 2-qonunlarini tushuntiring.
- R, L, C elementlaridagi kuchlanish va tokning matematik ifodasini yozing.
- RLC ketma-ket konturi uchun differensial tenglamani qanday tuzish mumkin?
- Rezonans nima va qachon yuzaga keladi?