4-mavzu: To‘la differensial tenglamalar va unga keladigan tenglamalar

Reja:

- To‘la differensial tenglamaning fizik va geometrik mazmuni
- To‘la differensial tenglama va uni tekshirish sharti
- To‘la differensial tenglama
- Misollar yechish

To‘la differensial tenglama

Ta’rif. Agar

$$M(x,y)dx + N(x,y)dy = 0 \qquad (1)$$

tenglamada $M(x,y)$ va $N(x,y)$ funksiyalar uzluksiz, differensiallanuvchi funksiyalar bo‘lib, bular uchun

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \qquad (2)$$

munosabat o‘rinli bo‘lsa, (1) tenglama to‘la differensialli tenglama deyiladi, bunda

$\frac{\partial M}{\partial y}$ va $\frac{\partial N}{\partial x}$ funksiyalar biror sohada uzluksiz funksiyalardir.

To‘la differensialli tenglamalarni integrallash.

Agar (1) tenglamaning chap tomoni to‘la differensial bo‘lsa, u holda (2) shartning bajarilishini va aksincha, (2) shart bajarilsa, (1) tenglamaning chap tomoni biror $u(x, y)$ funksiyaning to‘la differensiali bo‘lishini isbotlaymiz, yoki boshqacha aytganda (1) tenglamaning ko‘rinishi

$$du(x, y) = 0 \qquad (3)$$

bo‘lishini va demak, uning umumiy integrali

$$u(x, y) = C$$

bo‘lishini ko‘rsatamiz.

Dastlab, (1) tengl amaning chap tomonini biror $u(x, y)$ funksiyaning to‘la differensiali deb faraz qilaylik, ya’ni

$$M(x, y)dx + N(x, y)dy = du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy;$$

bu holda

$$M = \frac{\partial u}{\partial x}; \quad N = \frac{\partial u}{\partial y} \qquad (4)$$

Birinchi munosabatni y bo‘yicha, ikkinchi munosabatni esa x bo‘yicha, differensiallab,

$$\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial x \partial y}; \qquad \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial y \partial x}$$

tenglikni hosil qilamiz. Ikkinchi tartibli hosilalar uzluksiz deb faraz qilsak,

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

bo‘ladi, ya’ni (2) tenglik (1) tenglamaning chap tomoni biror $u(x, y)$

tenglamaning to‘la differensiali bo‘lishining zaruriy shartidan iboratdir. Bu shartning yetarli shart bo‘lishini ham, ya’ni (2) tenglik bajarilganda (1) tenglamaning chap tomoni biror

$u(x, y)$ funksiyaning to‘la differensiali bo‘lishini ko‘rsatamiz.

$$\frac{\partial u}{\partial x} = M(x, y)$$

munosabatdan

$$u = \int_{x_0}^{x} M(x, y)dx + \varphi(y)$$

ni topamiz, bunda x_0 yechim mavjud bo‘lgan sohadagi ixtiyoriy nuqtaning absissasi.

x bo‘yicha integrallashda y ni o‘zgarmas miqdor deb hisoblaymiz va shuning uchun integrallashda hosil bo‘lgan ixtiyoriy o‘zgarmas miqdor y ga bog’liq bo‘lishi mumkin.

$\varphi(y)$ ni (4) munosabatlardan ikkinchisi bajarilladigan qilib tanlab olamiz. Buning uchun keying tenglamaning ikkala tomonini

y bo‘yicha differensiallaymiz va natijani $N(x, y)$ ga tenglaymiz:

$$\frac{\partial u}{\partial y} = \int_{x_0}^{x} \frac{\partial M}{\partial y} dx + \varphi'(y) = N(x, y);$$

ammo $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ bo‘lgani uchun quyidagilarni yoza olamiz:

$$\int_{x_0}^{x} \frac{\partial N}{\partial x} dx + \varphi'(y) = N, \quad ya'ni \quad N(x, y)\big|_{x_0}^{x} + \varphi'(y) = N(x, y),$$

yoki

$$N(x,y) - N(x_0,y) + \varphi'(y) = N(x,y).$$

demak,

$$\varphi'(y) = N(x_0,y),$$

Yoki

$$\varphi(y) = \int_{y_0}^{y} N(x_0,y)dy + C_1.$$

shunday qilib, $u(x,y)$ funksiya

$$u = \int_{x_0}^{x} M(x,y)dx + \int_{y_0}^{y} N(x_0,y)dy + C_1$$

Ko‘rinishda bo‘ladi. Bunda $P(x_0, y_0)$ nuqta atrofida (1) differensial tenglamaning yechimi mavjud bo‘lgan nuqtadir.

Bu ifodani ixtiyoriy C o‘zgarmas miqdorga tenglab, (1) tenglamaning umumiy integralini hosil qilamiz:

$$\int_{x_0}^{x} M(x, y)dx + \int_{y_0}^{y} N(x_0, y)dy = C. \qquad (5)$$

1-misol.

$$\frac{2x}{y^3}dx + \frac{y^2 - 3x^2}{y^4}dy = 0$$

Tenglama berilgan. Bu tenglama to‘la differensialli tenglama bo‘lish yo bo‘lmasligini tekshiramiz.

Bu yerda

$$M = \frac{2x}{y^3}; \quad N = \frac{y^2 - 3x^2}{y^4}$$

Deb olamiz, bu holda

$$\frac{\partial M}{\partial y} = -\frac{6x}{y^4}; \quad \frac{\partial N}{\partial x} = -\frac{6x}{y^4}$$

$y \neq 0$ bo‘lganda (2) shart bajariladi. Demak, berilgan tenglamaning chap tomoni biror noma’lum $u(x,y)$ funksiyaning to‘liq differensiali bo‘ladi. Bu funksiyani topamiz. $\frac{\partial u}{\partial x} = \frac{2x}{y^3}$ bo‘lgani sababli

$$u = \int \frac{2x}{y^3} dx + \varphi(y) = \frac{x^2}{y^3} + \varphi(y)^3$$

Bunda $\varphi(y)$ funksiya y ning hozircha noma’lum bo‘lgan funksiyasi. Bu munosabatni y bo‘yicha differensiallab va

$$\frac{\partial u}{\partial x} = N = \frac{y^2 - 3x^2}{y^4}$$

tenglikni e’tiborga olib,

$$-\frac{3x^2}{y^4} + \varphi'(y) = \frac{y^2 - 3x^2}{y^4}$$

Bo‘lishini topamiz. Demak,

$$\varphi'(y) = \frac{1}{y^2}, \quad \varphi(y) = -\frac{1}{y} + C_1, \quad u(x, y) = \frac{x^2}{y^3} - \frac{1}{y} + C_1$$

Shunday qilib,

$$\frac{x^2}{y^3} - \frac{1}{y} = C$$

dastlabki tenglamaning umumiy integralidir.